Exact and Asymptotic Expressions of the Lift Slope Coefficient of an Elliptic Wing

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Introduction

A NEW analytical treatment has recently been presented for the classical lifting surface problem of an elliptic planform in incompressible steady[1] and unsteady[2] flow. Based on the expansion of the linearized acceleration potential in a series of ellipsoidal harmonics, the analysis led to very simple closed-form expressions for the aerodynamic coefficients in terms of the arbitrary aspect ratio. These exact expressions can be easily expanded into asymptotic forms to any order of the aspect ratio or its inverse for both high and low aspect ratios, respectively. The purpose of this Note is to obtain these asymptotic expressions and to compare them with other existing approximations.

The Exact Solution

We reproduce here only some of the final results of the analysis; for details, see Refs. 1 and 2.

The steady lift slope coefficient of an elliptic wing is[1]

$$\frac{C_L}{\alpha}=4\Big/\left\{k+\frac{E^2(h)}{k+(\arcsin h)/h}\right\},\qquad A\geq\frac{4}{\pi}$$

$$\frac{C_L}{\alpha}=4k\Big/\left\{1+\frac{E^2(h)}{1+(k^2/h)\log[(h+1)/k]}\right\},\quad A\leq\frac{4}{\pi}\tag{1}$$

The starting lift slope coefficient of an elliptic wing impulsively accelerated from rest is given by[2]

$$\frac{C_{LS}}{\alpha}=\frac{2[k+(\arcsin h)/h]}{E^2(h)},\qquad A\geq\frac{4}{\pi}$$

$$\frac{C_{LS}}{\alpha}=\frac{2\{k+(k^3/h)\log[(h+1)/k]\}}{E^2(h)},\quad A\leq\frac{4}{\pi}\tag{2}$$

where h denotes the eccentricity of the elliptic planform, $k=(1-h^2)^{1/2}$, and $E(h)$ is the complete elliptic integral of the second kind. The aspect ratio A is related to k by $A=4/(\pi k)$ for $A\geq 4/\pi$ and by $A=4k/\pi$ for $A\leq 4/\pi$. Both ranges of high and low aspect ratios correspond to $k\ll 1$.

Since the above expressions are valid in the whole range of aspect ratios between zero and infinity, their asymptotic expansions to any order of A or $1/A$ may be useful in testing the accuracy of corresponding expansions resulting from various approximate schemes.

Asymptotic Expresssions

Steady Lift

By using the following expansions for $k\ll 1$:

$$E(h)=1+\frac{1}{2}\left(\log\frac{4}{k}-\frac{1}{2}\right)k^2+\frac{3}{16}\left(\log\frac{4}{k}-\frac{13}{12}\right)k^4+\ldots$$

$$\arcsin h/h=\frac{\pi}{2}-k+\frac{\pi}{4}k^2-\frac{2}{3}k^3+\frac{3\pi}{16}k^4+\ldots\tag{3}$$

we may easily obtain the asymptotic expressions for the steady lift slope to any desired order of k. Hence, for high aspect ratios, the fourth-order expansion may be written as

$$\frac{C_L}{\alpha}=2\pi\Big/\left\{1+2aA^{-1}+\frac{16}{\pi^2}[b\log(\pi A)-c]A^{-2}+dA^{-3}+eA^{-4}+\ldots\right\}\tag{4}$$

where

$$a=1,\quad b=1,\quad c=1,\quad d=256/(3\pi^4)$$

$$e=\frac{64}{\pi^4}\left[\log^2(\pi A)-\frac{3}{2}\log(\pi A)-\frac{7}{8}\right]\tag{5}$$

The first two terms in the denominator of Eq. (4) correspond to the first-order well-known Prandtl lifting line theory. The next term is similar to the one obtained by Van Dyke[3] in his higher-order lifting line theory, but with a different value of the constant c [$c=7/8$ in the original version,[3] then $c=9/8$ after a correction due to Kerney[4] that is included in Eq. (N30) of Ref. 5]. Kida and Miyai[6] and Kida[7] have shown that the correct value is $c=1$, which is confirmed again by the present analysis. Comparisons with values of the constants appearing in various previous asymptotic expressions are given in Table 1.

It should be noted that Krienes[8] derived his expression from his lifting-surface analysis. Kida refined Krienes' analysis in Ref. 9 and later obtained the corresponding third-order asymptotic expression in Ref. 7. Both the Krienes and Kida lifting-surface analyses required inversion of an infinite set of linear equations (a different set for each A) in order to derive the lift slope in contrast with the explicit equations (1) obtained in Ref. 1. The values referred to as Helmbold's are described by Jordan[10] to result from Helmbold's approximation analysis.[11] Jones'[12] values may be easily derived from his remarkably simple modification of the Prandtl lifting-line result, which is given by

$$\frac{C_L}{\alpha}\cong\frac{2\pi}{E(h)+2/A}\tag{6}$$

By using the expansion

$$\log\left(\frac{1-h}{k}\right)=\log\left(\frac{2}{k}\right)-\left(\frac{k^2}{4}\right)+\ldots\tag{7}$$

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the following expansion is easily obtained from Eqs. (1) for $A \ll 1$:

$$\frac{C_L}{\alpha} = \frac{(\pi/2)/A}{1 + (\log 2 - 0.5)A^2 + \mathcal{O}(A^4)} \tag{8}$$

The higher-order terms in Eq. (8) can be derived without difficulty. Equation (8) should be considered as a higher-order approximation of the slender-wing theory, the first-order approximation of which was first given by Jones.[13] Equation (8) provides a fair approximation over a wide range of A. For example, for the circular planform ($A = 4/\pi$) it gives 1.8239, which is about 1.8% higher than the exact value and very close to the value 1.82 usually quoted as the "exact" result of Kinner's[14] original analysis of the circular wing.

Unsteady Lift

By using Eqs. (3) and (8), the following asymptotic expressions are obtained for the starting lift slope of an impulsively accelerated wing:

$$\frac{C_{LS}}{\alpha} = \pi \Bigg/ \left\{ 1 + \frac{16}{\pi^4}[\log(\pi A) - 1]A^{-2} + \frac{256}{3\pi^4} A^{-3} + \mathcal{O}(A^{-4}) \right\}, \qquad A \gg 1 \tag{9}$$

Equation (9) should be compared to Jones[15] approximate expression $C_{LS}/\alpha = \pi/E(h)$, which has the following asymptotic expansion:

$$\frac{C_{LS}}{\alpha} = \pi \Bigg/ \left\{ 1 + \frac{8}{\pi^2}[\log(\pi A) - 0.5]A^{-2} + \mathcal{O}(A^{-4}) \right\} \tag{10}$$

The corresponding expansion of Eqs. (2) for $A \ll 1$ is

$$\frac{C_{LS}}{\alpha} = \frac{\pi}{2} A - \frac{\pi^2}{8}\left(\log \frac{16}{\pi A} - \frac{1}{2}\right)A^2 - \frac{\pi^3}{32}(\log 2 - 0.5)A^3 + \dots \tag{11}$$

Table 1 Comparison between the coefficients in different asymptotic expressions of the type of Eq. (4)

	a	b	c	d	e
Prandtl	1	—	—	—	—
Van Dyke[5]	1	1	9/8	—	—
Kida and Miyai[6]	1	1	1	—	—
Kida[7]	1	1	1	$(16/\pi^2) \times [\log(\pi A) - 1.5]$	—
Kriens[8]	63/64	1	7/4	—	—
Helmbold[11]	1	1/2	−1/4	—	—
Jones[12]	1	1/2	1/2	—	—
Present [Eq. (4)]	1	1	1	$256/(3\pi^4)$	$\frac{64}{\pi^4}[\log^2(\pi A) - 1.5\log(\pi A) - 7/8]$

Equation (11) shows that, in the limit $A \to 0$, the starting lift tends to the steady value $(\pi/2)/A$ and is always lower than the steady lift for any finite A, as may be observed by directly examining the exact equations (2).

Conclusion

The closed-form expressions for the lift slope of the elliptic wing may be asymptotically expanded for both low and high aspect ratios to any order of the aspect ratio or its inverse. The asymptotic expansion for high A is identical up to second order to the result of Kida and Miyai,[6] but differs in the third-order term from Kida's expansion.[7] The corresponding expansions for low A, as well as the expression for the starting lift in unsteady motion, may be useful for examining various results of approximate methods.

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References

[1]Hauptman, A. and Miloh, T., "On the Exact Solution of the Linearized Lifting Surface Problem of an Elliptic Wing," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 39, Pt. 1, 1986, pp. 41-66.

[2]Hauptman, A. and Miloh, T., "Aerodynamic Coefficients of a Thin Wing with Elliptic Planform in Unsteady Motion," *AIAA Journal*, Vol. 25, 1987, pp. 769-774.

[3]Van Dyke, M., "Lifting Line Theory as a Singular Perturbation Problem," *Journal of Applied Mathematics and Mechanics*, Vol. 29, Pt. 1, 1964, pp. 90-102.

[4]Kerney, K. P., "A Correction to 'Lifting-Line Theory as a Singular Perturbation Problem'," *AIAA Journal*, Vol. 10, 1972, pp. 1683-1684.

[5]Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Parabolic Press, 1978.

[6]Kida, T. and Miyai, Y., "An Alternative Treatment of Lifting Line Theory as a Perturbation Problem," *ZAMP*, Vol. 29, 1978, pp. 591-607.

[7]Kida, T., "An Asymptotic Expression of Lift-Slope of Elliptic Wing with High Aspect Ratio," *ZAMM*, Vol. 62, 1982, pp. 491-493.

[8]Krienes, K., "Die elliptische Tragflache auf potential-theoretischer Grundlage," *ZAMM*, Vol. 20, 1940, pp. 65-88 (English translation, NACA TM 1324, 1953).

[9]Kida, T., "A Theoretical Treatment of Lifting-Surface Theory of an Elliptic Wing, *ZAMM*, Vol. 60, 1980, pp. 645-651.

[10]Jordan, P. F., "Remarks on Applied Subsonic Lifting-Surface Theory," *Jahrbuch 1967 de WGLR*, 1967, pp. 192-210.

[11]Helmbold, H. B., "Der unverwundene Ellipsenflugel als tragende Flache," *Jahrbuch 1942 der Deutsch Luftfahrtforsch*, 1942, pp. I111-I113.

[12]Jones, R. T., "Theoretical Correction for the Lift of Elliptic Wings," *Journal of the Aeronautical Sciences*, Vol. 9, 1940, pp. 8-10.

[13]Jones, R. T., "Properties of Low-Aspect-Ratio Pointed Wings at Speeds below and above the Speed of Sound," NACA Rept. 835, 1946.

[14]Kinner, W., "Die Kreisformige Tragflache auf potential-theoretischer Grundlage," *Ingenieur Archiv.*, Vol. 8, 1937, pp. 47-80 (English translation: R.T.P. Translation No. 2345, issued by the Ministry of Aircraft Production).

[15]Jones, R. T., "The Unsteady Lift of a Wing of Finite Aspect Ratio," NACA Rept. 681, 1939.